DTP/00/12
February 2000

Complete Renormalization Group Improvement- Avoiding Factorization and Renormalization Scale Dependence in QCD Predictions

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Abstract

For moments of leptoproduction structure functions we show that all dependence on the renormalization and factorization scales disappears provided that all the ultraviolet logarithms involving the physical energy scale Q are completely resummed. The approach is closely related to Grunberg's method of Effective Charges. A direct and simple method for extracting $\Lambda_{\overline{MS}}$ from experimental data is advocated.

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1 Introduction

The problem of renormalization scheme dependence in QCD perturbation theory remains on obstacle to making precise tests of the theory. In a recent paper [1] one of us pointed out that the renormalization scale dependence of dimensionless physical QCD observables, depending on a single energy scale Q, can be avoided provided that all ultraviolet logarithms which build the physical energy dependence on Q are resummed. This was termed complete Renormalization Group (RG)-improvement in Ref.[1]. It was stressed that standard RG-improvement, as customarily applied with a Q-dependent scale $\mu = xQ$, omits an infinite subset of these logarithms. One should rather keep μ independent of Q, and then carefully resum to all-orders the RG-predictable ultraviolet logarithms. In this way all μ-dependence cancels between the renormalized coupling and the logarithms of μ contained in the coefficients, and the correct physical Q-dependence is built. At next-to-leading order (NLO) the result is identical to the Effective Charge approach of Grunberg [2, 3]. We wish to extend this argument to processes involving factorization of operator matrix elements and coefficient functions, where a factorization scale M arises in addition to the renormalization scale μ. We shall use the prototypical factorization problem of moments of leptoproduction structure functions as a specific example. We shall identify the logarithms of μ ,M , and Q which occur, and will show explicitly that on resumming all the ultraviolet logarithms the μ and M dependence disappears. We shall organize the paper so that we review the treatment of Ref.[1] whilst showing how it generalizes for the moment problem. We begin in Section 2 by giving some basic definitions for the moments of structure functions. Section 3 considers the dependence of the perturbative coefficients on the parameters which label the renormalization procedure in both cases. Section 4 deals with the complete RG-improvement of the structure function moments and identifies and resums the physical ultraviolet logarithms. Finally, in Section 5 we discuss a more straightforward way of motivating this approach, and consider how to directly extract $\Lambda_{\overline{MS}}$ from data. We also give our Conclusions.

2 Structure Function Moments

In the prototypical factorization problem of deep inelastic leptoproduction the n^{th} moment of a non-singlet structure function $F(x)$,

$$\mathcal{M}_n(Q) = \int_0^1 x^{n-2} F(x)\, dx \ , \tag{1}$$

can be factorized in the form

$$\mathcal{M}_n(Q) = < \mathcal{O}_n(M) > \mathcal{C}_n(Q, a(\mu), \mu, M) \ . \tag{2}$$

Here M is an arbitrary factorization scale and $a(\mu)$ is the RG-improved coupling $\alpha_s(\mu)/\pi$ defined at a renormalization scale μ. The operator matrix element $< \mathcal{O}_n(M) >$ has an M-dependence given by its anomalous dimension,

$$\frac{M}{< \mathcal{O} >} \frac{\partial < \mathcal{O} >}{\partial M} = \gamma_{\mathcal{O}}(a) = -da - d_1 a^2 - d_2 a^3 - d_3 a^4 + \ldots \tag{3}$$

For simplicity we shall from now on suppress the n-dependence of terms in equations, as we have done in Eq.(3). For a given moment d is independent of the factorization convention, whereas the higher d_i ,($i \geq 1$) depend on it. In Eq.(3) the coupling a is governed by the β-function equation

$$M \frac{\partial a}{\partial M} = \beta(a) = -ba^2(1 + ca + c_2 a^2 + c_3 a^3 + \ldots) \ . \tag{4}$$

Here $b = (33 - 2N_f)/6$, and $c = (153 - 19N_f)/12b$, are the first two coefficients of the beta-function for SU(3) QCD with N_f active flavours of quark. They are universal, whereas the subsequent coefficients $c_2, c_3, \ldots$ are scheme-dependent. Equation (3) can be integrated to [4, 5]

$$< \mathcal{O}(M) > = A \exp\left[\int_0^a \frac{\gamma(x)}{\beta(x)}\, dx - \int_0^\infty \frac{\gamma^{(1)}(x)}{\beta^{(2)}(x)}\, dx\right] \ , \tag{5}$$

where $\gamma^{(1)}$ and $\beta^{(2)}$ denote these functions truncated at one and two terms, respectively. The factor A is scheme-independent [5] and can be fitted to experimental data. The second integral in Eq.(5) is an infinite constant of

integration. In Eq.(2) $\mathcal{C}(Q, a(\mu), \mu, M)$ is the coefficent function and has the perturbation series

$$\mathcal{C}(Q, \tilde{a}, \mu, M) = 1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3 + \dots . \tag{6}$$

We shall use $\tilde{a}$ to stand for $a(\mu)$ and a for $a(M)$. After combining the integrals in Eq.(5) one obtains

$$\mathcal{M} = A\Big(\frac{ca}{1+ca}\Big)^{d/b} \exp(\mathcal{I}(a)) \; (1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3 + \dots) \; , \tag{7}$$

where $\mathcal{I}(a)$ is the finite integral

$$\mathcal{I}(a) = \int_0^a dx \; \frac{d_1 + (d_1 c + d_2 - dc_2)x + (d_3 + cd_2 - c_3 d)x^2 + \dots}{b(1+cx)(1 + cx + c_2x^2 + c_3x^3 + \dots)} \; , \tag{8}$$

which can be readily evaluated numerically. The coupling $a(\tau)$ itself, where $\tau \equiv b\ln(\mu/\tilde{\Lambda})$, is obtained as the solution of the transcendental equation [6]

$$\frac{1}{a} + c\ln\frac{ca}{1+ca} = \tau - \int_0^a dx \left[-\frac{1}{B(x)} + \frac{1}{x^2(1+cx)} \right] \; , \tag{9}$$

where $B(x) \equiv x^2(1 + cx + c_2x^2 + c_3x^3 + \dots)$.

3 RS and FS dependence of the coefficients

We first wish to parametrize the dependence of the r_n in the coefficient function on the renormalization scheme (RS) and factorization scheme (FS).

Recall first [6] that for the single scale case of a dimensionless observable $\mathcal{R}(Q)$ with perturbation series

$$\mathcal{R}(Q) = a + r_1a^2 + r_2a^3 + \dots + r_na^{n+1} + \dots \; , \tag{10}$$

the RS can be labelled by the non-universal coefficients of the beta-function c_2,c_3,...,and by τ, which can be traded as a parameter for r_1 since [2, 3, 6, 7]

$$\tau - r_1 = \rho_0(Q) \equiv b\ln(Q/\Lambda_{\mathcal{R}}) \; , \tag{11}$$

is an RS-invariant.Using the self-consistency of perturbation theory- that is that the difference between a N^{n}LO calculation (i.e up to and including

$r_n a^{n+1}$) performed with two different RS's is $O(a^{n+2})$, one can derive expressions for the partial derivatives of the perturbative coefficients with respect to the scheme parameters. For r_2 for instance one has [6]

$$\frac{\partial r_2}{\partial r_1} = 2r_1 + c,\ \frac{\partial r_2}{\partial c_2} = -1,\ \frac{\partial r_2}{\partial c_3} = 0, ... \tag{12}$$

on integration one finds

$$\begin{aligned} r_2(r_1, c_2) &= {r_1}^2 + c r_1 + X_2 - c_2 \\ r_3(r_1, c_2, c_3) &= {r_1}^3 + \frac{5}{2} c {r_1}^2 + (3X_2 - 2c_2) r_1 + X_3 - \frac{1}{2} c_3 \\ \vdots \quad & \quad \vdots\,. \end{aligned} \tag{13}$$

In general the structure is

$$r_n(r_1, c_2, \ldots, c_n) = \hat{r}_n(r_1, c_2, \ldots, c_{n-1}) + X_n - c_n/(n-1)\ , \tag{14}$$

where $\hat{r}_n$ is RG-predictable from a complete N^{n-1}LO calculation (i.e. $r_2, r_3, \ldots, r_n$, and $c_2, c_3, \ldots, c_n$ have been computed in some RS), and the X_n are Q-independent and RS-invariant constants of integration which are unknown unless a complete N^nLO calculation has been performed.

As we shall see the generalization to the moment problem is a dependence $r_n(\mu, M, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n)$ where the c_i label the RS and the d_i the FS. As before M,μ can be traded, in this case for $r_1(M)$ and $\tilde{r}_1 \equiv r_1(M = \mu)$. There will be analogous factorization and renormalization scheme (FRS) invariants ,X_n, which represent the RG-unpredictable parts of r_n. Expressions for the dependence of the coefficients on FRS parameters have been derived before in Refs.[4, 5, 8] , but there were some errors in Ref.[4], in particular the dependence of r_2 on c_2 was not recognized [5]. Partially differentiating Eq.(7) with respect to $\mu, M, c_2, c_3, d_1, d_2, d_3$, and demanding for consistency that it be $O(a^4)$, so that the coefficients of a, a^2 and a^3 vanish, one obtains analogous to Eqs.(12),

$$\begin{aligned} \mu \frac{\partial r_1}{\partial \mu} &= 0, \quad \mu \frac{\partial r_2}{\partial \mu} = r_1 b, \quad \mu \frac{\partial r_3}{\partial \mu} = 2 r_2 b + r_1 b c\ , \\ M \frac{\partial r_1}{\partial M} &= d, \quad M \frac{\partial r_2}{\partial M} = d_1 + d r_1 - dL\ , \end{aligned}$$

$$
\begin{aligned}
M\frac{\partial r_3}{\partial M} &= d_2 + d_1 r_1 + d r_2 - d r_1 L - 2d_1 L - dL^2, \\
\frac{\partial r_1}{\partial d_1} &= -\frac{1}{b}\,, \frac{\partial r_2}{\partial d_1} = \frac{c}{2b} - \frac{L}{b} - \frac{r_1}{b}, \\
\frac{\partial r_3}{\partial d_1} &= \frac{c r_1}{2b} - \frac{c^2}{3b} + \frac{(c-r_1)}{b}L - \frac{r_2}{b} + \frac{c_2}{3b} - \frac{L^2}{b}, \\
\frac{\partial r_1}{\partial d_2} &= 0, \quad \frac{\partial r_2}{\partial d_2} = -\frac{1}{2b}, \quad \frac{\partial r_3}{\partial d_2} = \frac{c}{3b} - \frac{L}{b} - \frac{r_1}{2b}\ , \\
\frac{\partial r_1}{\partial d_3} &= 0, \quad \frac{\partial r_2}{\partial d_3} = 0, \quad \frac{\partial r_3}{\partial d_3} = -\frac{1}{3b}\ , \\
\frac{\partial r_1}{\partial c_2} &= 0, \quad \frac{\partial r_2}{\partial c_2} = \frac{3d}{2b}, \quad \frac{\partial r_3}{\partial c_2} = \frac{4d_1}{3b} + 3\frac{dL}{b} + 3\frac{d r_1}{2b} - r_1 - 5\frac{cd}{3b}\ , \\
\frac{\partial r_1}{\partial c_3} &= 0, \quad \frac{\partial r_2}{\partial c_3} = 0, \quad \frac{\partial r_3}{\partial c_3} = \frac{5d}{6b}\ . \qquad (15)
\end{aligned}
$$

Here we have defined for convenience $L \equiv b\ln(M/\mu)$. Consistently integrating the partial derivatives of r_1 yields

$$
r_1 = \frac{d}{b}\tau_M - \frac{d_1}{b} - X_1(Q)\ , \qquad (16)
$$

where $\tau_M \equiv b\ln(M/\tilde{\Lambda})$ and $X_1(Q)$ is an FRS-invariant, analogous to $\rho_0(Q)$ for the single scale problem defined in Eq.(11). Exactly analogous to $\Lambda_{\mathcal{R}}$, for the moment problem one can define an FRS-invariant $\Lambda_{\mathcal{M}}$ so that

$$
\frac{d}{b}\tau_M - \frac{d_1}{b} - r_1 = X_1(Q) \equiv d\ln\left(\frac{Q}{\Lambda_{\mathcal{M}}}\right)\ . \qquad (17)
$$

Consistently integrating the remaining partial derivatives and using Eq.(16) to recast the M and μ dependence in terms of r_1 and $\tilde{r}_1$, one obtains the explicit dependence of r_2 and r_3 on the FRS parameters $r_1, \tilde{r}_1, d_1, d_2, d_3, c_2, c_3$,

$$
\begin{aligned}
r_2 &= \left(\frac{1}{2} - \frac{b}{2d}\right){r_1}^2 + \frac{b}{d}r_1\tilde{r}_1 + \frac{c d_1}{2b} - \frac{d_2}{2b} - \frac{d c_2}{2b} + X_2 \\
r_3 &= \left(\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2}\right)\frac{{r_1}^3}{3} + \left(-\frac{b^2}{d^2} + \frac{b}{d}\right){r_1}^2\tilde{r}_1 + \left(\frac{bc}{d} + \frac{2bd_1}{d^2}\right)r_1\tilde{r}_1 \\
&\quad + \left(-\frac{bc}{2d} - \frac{bd_1}{d^2} + \frac{d_1}{d}\right){r_1}^2 + \left(-\frac{dc_2}{2b} + \frac{cd_1}{2b} + X_2 + \frac{{d_1}^2}{2db} + \frac{d_2}{d} - \frac{d_2}{2b} - c_2\right)r_1
\end{aligned}
$$

$$
\begin{aligned}
&+(\frac{{d_1}^2}{d^2}-\frac{d_2}{d}+\frac{cd_1}{d}+\frac{2bX_2}{d})\tilde{r}_1+\frac{b^2}{d^2}r_1\tilde{r}_1^2+(-\frac{d_1c^2}{3b}+\frac{2d_1X_2}{d}\\
&+\frac{{d_1}^3}{3bd^2}+\frac{dcc_2}{3b}+\frac{c{d_1}^2}{2db}+\frac{d_3}{3d}-\frac{dc_3}{6b}-\frac{2d_1c_2}{3b}+\frac{d_2c}{3b}+X_3)\\
&\vdots \quad \vdots ,
\end{aligned} \tag{18}
$$

analogous to Eqs.(13) in the single scale case. Notice that we could equally use r_1 and L as parameters instead of r_1 and $\tilde{r}_1$, since $L=(b/d)(r_1-\tilde{r}_1)$. As in the single scale case there are constants of integration X_n representing the RG-unpredictable part of r_n. They are Q-independent and FRS-invariant.

In the single scale case parametrizing the RS-dependence using $r_1, c_2, c_3, \ldots$ means that given a complete N^{n}LO calculation $X_2, X_3, \ldots, X_n$ will be known. Using Eqs.(13) to sum to all-orders the RG-predictable terms, i.e. those *not* involving $X_{n+1}, X_{n+2}, \ldots$, with coupling $a(r_1, c_2, c_3, \ldots)$ is equivalent to N^{n}LO perturbation theory in the scheme with $r_1=c_2=c_3=\ldots=0$, and yields the sum

$$
\mathcal{R}^{(n)}=a_0+X_2{a_0}^2+X_3{a_0}^3+\ldots+X_n{a_0}^n\ , \tag{19}
$$

where $a_0\equiv a(0,0,0,\ldots)$ is the coupling in this scheme. From Eqs.(9) and (11) it satisfies

$$
\frac{1}{a_0}+c\ln\left(\frac{ca_0}{1+ca_0}\right)=b\ln\left(\frac{Q}{\Lambda_{\mathcal{R}}}\right)\ . \tag{20}
$$

In fact the solution of this transcendental equation can be written in closed form in terms of the Lambert W-function [9, 10], defined implicitly by $W(z)\exp(W(z))=z$,

$$
\begin{aligned}
a_0 &= -\frac{1}{c[1+W(z(Q))]}\\
z(Q) &\equiv -\frac{1}{e}\left(\frac{Q}{\Lambda_{\mathcal{R}}}\right)^{-b/c}\ .
\end{aligned} \tag{21}
$$

A similar expansion to Eq.(19), but motivated differently, has been suggested in Ref.[11].

In the moment problem by an exactly similar argument, with the chosen parametrization of FRS, given a complete N^{n}LO calculation (i.e. a calculation of $r_1, r_2, \ldots, r_n$ and the $d_1, d_2, \ldots, d_n$ and $c_2, c_3, \ldots, c_n$ in some FRS) the

invariants $X_2, X_3, \ldots, X_n$ will be known. Using Eqs.(18) to sum to all-orders the RG-predictable terms not involving $X_{n+1}, X_{n+2}, \ldots$, will be equivalent to working with an FRS in which all the FRS parameters are set to zero. $\tilde{r}_1 = 0$ means that $\mu = M$. Setting $r_1 = 0$, $d_1 = 0$ in Eq.(17) yields $\tau_M = b\ln(Q/\Lambda_{\mathcal{M}})$, so that $a = \tilde{a} = a_0$, given by Eq.(21) with $\Lambda_{\mathcal{R}}$ replaced by $\Lambda_{\mathcal{M}}$. Further, with $c_i = d_i = 0$ the integral $\mathcal{I}(a)$ in Eq.(8) vanishes, so that finally the sum of all RG-predictable terms for the moment problem at N^{n}LO will be

$$\mathcal{M} = A\left(\frac{ca_0}{1+ca_0}\right)^{d/b}(1 + X_2{a_0}^2 + X_3{a_0}^3 + \ldots + X_n{a_0}^n) \; , \tag{22}$$

with an extremely similar structure to the single scale case in Eq.(19). Substituting for a_0 in terms of the Lambert W-function using Eq.(21) we then obtain

$$\begin{aligned} \mathcal{M} &= A[-W(z(Q))]^{b/d}(1 + X_2{a_0}^2 + \ldots) \\ z(Q) &\equiv -\frac{1}{e}\left(\frac{Q}{\Lambda_{\mathcal{M}}}\right)^{-b/c} . \end{aligned} \tag{23}$$

So that moments of structure functions have a Q-dependence naturally involving a power of the Lambert W-function.

As stressed in Ref.[1] the result of resumming all RG-predictable terms depends on the chosen parametrization of RS. By simply translating the parameters to a new set $\breve{r}_1 = r_1 - \overline{r}_1, \breve{c}_2 = c_2 - \overline{c}_2, \ldots$ etc., where the barred quantities are constants, one finds corresponding new constants of integration $\breve{X}_n$. The result of resumming all RG-predictable terms with this new parametrization then corresponds to standard fixed-order perturbation theory in the RS with $r_1 = \overline{r}_1, c_2 = \overline{c}_2, \ldots$, or equivalently with $\breve{r}_1 = \breve{c}_2 = \breve{c}_3 = \ldots = 0$. The key point is that r_1 has a special status since it contains the ultraviolet (UV) logarithms which build the physical Q-dependence of $\mathcal{R}(Q)$. Standard RG-improvement corresponds to shifting the parameter r_1, in which case the resulting constants of integration $\breve{X}_n$ contain physical UV logarithms which are not all resummed. Thus r_1 should be used as the parameter. An exactly similar statement holds for r_1 and $\tilde{r}_1$ in the moment problem. We shall identify the UV logarithms and show how their complete resummation builds the correct physical Q-dependence in the next section.

We shall refer to the expansions in Eqs.(19) and (22) as Complete RG-improved (CORGI) results. Whilst the parameters implicitly containing the UV logarithms do have a special status, the remaining dimensionless parameters c_i and d_i can be reparametrized as one pleases. As an example, in the Effective Charge approach of Grunberg [2, 3] one chooses $\overline{c}_2, \overline{c}_3, \ldots, \overline{c}_n$ so that $\breve{X}_2, \breve{X}_3, \ldots, \breve{X}_n$ are all zero at N^{n}LO, corresponding to $r_1 = r_2 = \ldots = r_n = 0$, and this is *a priori* equally reasonable. In the moment problem one can correspondingly choose the $\overline{c}_i$ and $\overline{d}_i$ so that at N^{n}LO the $\breve{X}_i$ all vanish and $r_1 = r_2 = \ldots = r_n = 0$. If one further demands that the integral $\mathcal{I}(a)$ in Eq.(8) vanishes order-by-order in a a unique FRS is selected in which moments have the form

$$\mathcal{M} = A\left(\frac{c\hat{\mathcal{R}}}{1+c\hat{\mathcal{R}}}\right)^{d/b} . \tag{24}$$

Where $\hat{\mathcal{R}}$ is an effective charge which has a perturbation series of the form,

$$\hat{\mathcal{R}} = a + \hat{r}_1 a^2 + \hat{r}_2 + \ldots + \hat{r}_n a^{n+1} + \ldots . \tag{25}$$

This is similar to Grunberg's proposal [3] to associate an effective charge with $\mathcal{M}$ so that $\mathcal{M} = A(c\hat{\mathcal{R}})^{d/b}$. The $\hat{r}_i$ are built from the c_i, d_i, M and μ,and are RS-dependent, but FS-independent. Effectively $\hat{\mathcal{R}}$ can be RG-improved as in the single scale case. We have for instance

$$\hat{r}_1 = b\ln(\mu/\tilde{\Lambda}) - b\ln(M/\tilde{\Lambda}) - \frac{b}{d}r_1 + d_1/d = \tau - X_1(Q) , \tag{26}$$

where we have used Eq.(17) . Comparing with Eq.(11) we see that treating $\hat{\mathcal{R}}$ as a single scale problem we have $\rho_0(Q) = X_1(Q)$. This further implies that $\Lambda_{\hat{\mathcal{R}}} = \Lambda_{\mathcal{M}}$ and so the corresponding CORGI couplings are identical. The CORGI expansion for $\hat{\mathcal{R}}$ will be of the form

$$\hat{\mathcal{R}} = a_0 + \hat{X}_2 {a_0}^2 + \hat{X}_3 {a_0}^3 + \ldots . \tag{27}$$

Inserting this result in Eq.(24) and re-expanding in a_0 will reproduce the CORGI expansion in Eq.(22).

4 Complete RG-improvement

In the single scale case using Eq.(11) one can write

$$r_1 = b\left(\ln\frac{\mu}{\tilde{\Lambda}} - \ln\frac{Q}{\Lambda_{\mathcal{R}}}\right) . \tag{28}$$

The first μ-dependent logarithm depends on the RS, whereas the second Q-dependent UV logarithm will generate the physical Q-dependence and is RS-invariant. If one makes the simplification that $c = 0$ and sets $c_2 = c_3 = \ldots = 0$, then the coupling is given by

$$a(\mu) = 1/b\ln(\frac{\mu}{\tilde{\Lambda}}) . \tag{29}$$

The sum to all-orders of the RG-predictable terms from Eqs.(13), given a NLO calculation of r_1, simplifies to a geometric progression,

$$\mathcal{R} = a + r_1 a + {r_1}^2 a^3 + \ldots + {r_1}^n a^{n+1} + \ldots . \tag{30}$$

The idea of complete RG-improvement is that dimensionful renormalization scales, in this case μ, should be held strictly independent of the physical energy scale Q on which $\mathcal{R}(Q)$ depends. In this way the Q-dependence is built entirely by the "physical" UV logarithms $b\ln(Q/\Lambda_{\mathcal{R}})$ contained in r_1, and the convention-dependent logarithms of μ cancel between $a(\mu)$ and $r_1(\mu)$, when the all-orders sum in Eq.(30) is evaluated. The conventional fixed-order NLO truncation $\mathcal{R} = a(\mu) + r_1(\mu)a(\mu)^2$, only makes sense if $\mu = xQ$, but then the resulting Q-dependence involves the arbitrary parameter x. In contrast using Eqs.(28),(29) and summing the geometric progression in Eq.(30) gives,

$$\mathcal{R}(Q) \approx a(\mu)/\left[1 - \left(b\ln\frac{\mu}{\tilde{\Lambda}} - b\ln\frac{Q}{\Lambda_{\mathcal{R}}}\right)a(\mu)\right] = 1/b\ln(Q/\Lambda_{\mathcal{R}}) , \tag{31}$$

correctly reproducing the large-Q behaviour of $\mathcal{R}(Q)$,

$$\mathcal{R}(Q) \approx 1/b\ln(Q/\Lambda_{\mathcal{R}}) + O(1/b\ln(Q/\Lambda_{\mathcal{R}})^3 . \tag{32}$$

In the moment problem the analogous UV logarithm is $b\ln(Q/\Lambda_{\mathcal{M}})$ introduced in Eq.(17), and analogous to Eq.(28) we will have

$$r_1 = d\left(\ln\frac{M}{\tilde{\Lambda}} - \ln\frac{Q}{\Lambda_{\mathcal{M}}}\right) - \frac{d_1}{b} . \tag{33}$$

Given a NLO calculation of r_1 we wish to see how the physical Q-dependence of $\mathcal{M}(Q)$ arises on resumming to all-orders the UV logarithms contained in the RG-predictable terms from Eqs.(18). If we make similar approximations, so that $c=0$ and the d_i and c_i are set to zero, then

$$\mathcal{M} = A(ca(M))^{d/b}(1 + r_1 a(\mu) + r_2 a(\mu)^2 + \ldots)\ . \tag{34}$$

We retain the overall factor of $c^{d/b}$. The task is then to show that on resumming the RG-predictable terms in the coefficient function to all-orders the $\ln(M/\tilde{\Lambda})$ and $\ln(\mu/\tilde{\Lambda})$ contained in r_1 and $\tilde{r_1}$ cancel with those in the couplings $a(M)$ and $a(\mu)$ to yield the physical Q-dependence

$$\mathcal{M}(Q) \approx A c^{d/b}(1/b\ln(Q/\Lambda_{\mathcal{M}}))^{d/b}(1 + O(1/\ln(Q/\Lambda_{\mathcal{M}}))^2)\ . \tag{35}$$

Again, the complete RG-improvement summing over all UV logarithms is forced on one if μ and M are held independent of Q.

The algebraic structure of the resummation of RG-predictable terms for the moment problem is considerably more complicated than the geometric progression of Eq.(30) encountered in the single scale case. With the simplifications $c=0, c_i=0, d_i=0$ the first two RG-predictable coefficients from Eqs(18) are

$$r_2 = (\frac{1}{2} - \frac{b}{2d}){r_1}^2 + \frac{b}{d} r_1 \tilde{r_1} \tag{36}$$

$$r_3 = (\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2})\frac{{r_1}^3}{3} + (\frac{-b^2}{d^2} + \frac{b}{d}){r_1}^2\tilde{r_1} + \frac{b^2}{d^2} r_1 {\tilde{r_1}}^2 \tag{37}$$

Suitably generalizing the partial derivatives in Eqs.(15) one can arrive at a general form for the RG-predictable terms. It is useful to arrange them in columns,

$$\begin{pmatrix}
r_1 \to (\frac{b}{d}\tilde{r_1})^0 r_1 \tilde{a} & 0 & 0 & \ldots \\
r_2 \to (\frac{b}{d}\tilde{r_1})^1 r_1 \tilde{a}^2 & (1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^2 & 0 & \ldots \\
r_3 \to (\frac{b}{d}\tilde{r_1})^2 r_1 \tilde{a}^3 & 2(\frac{b}{d}\tilde{r_1})(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^3 & (1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^3 & \ldots \\
r_4 \to (\frac{b}{d}\tilde{r_1})^3 r_1 \tilde{a}^4 & 3(\frac{b}{d}\tilde{r_1})^2(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^4 & 3(\frac{b}{d}\tilde{r_1})(1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^4 & \ldots \\
r_5 \to (\frac{b}{d}\tilde{r_1})^4 r_1 \tilde{a}^5 & 4(\frac{b}{d}\tilde{r_1})^3(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^5 & 6(\frac{b}{d}\tilde{r_1})^2(1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^5 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{38}$$

The idea will be to resum each column separately. Denoting the sum of the mth column by S_m, one finds

$$\begin{aligned} S_1 &= r_1\tilde{a} + (\frac{b}{d}\tilde{r_1})r_1\tilde{a}^2 + (\frac{b}{d}\tilde{r_1})^2 r_1\tilde{a}^3 + (\frac{b}{d}\tilde{r_1})^3 r_1\tilde{a}^4 + (\frac{b}{d}\tilde{r_1})^4 r_1\tilde{a}^5 + \ldots \\ &= r_1\tilde{a}[1 + (\frac{b}{d}\tilde{r_1}\tilde{a}) + (\frac{b}{d}\tilde{r_1}\tilde{a})^2 + (\frac{b}{d}\tilde{r_1}\tilde{a})^3 + (\frac{b}{d}\tilde{r_1}\tilde{a})^4 + \ldots] \\ &= r_1\tilde{a}(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-1} \end{aligned} \tag{39}$$

Careful examination of the pattern of terms in Eq.(38) leads to the general result for S_m for $m > 1$,

$$S_m = (-1)^{2m-1}(\frac{b}{d} - 1)(\frac{b}{d} - \frac{1}{2})(\frac{b}{d} - \frac{1}{3}) + \ldots + (\frac{b}{d} - \frac{1}{m-1})\frac{{S_1}^m}{m} \tag{40}$$

Finally the resummed RG-predictable terms in the coefficient function will follow from $\mathcal{C} = 1 + S_1 + S_2 + S_3 + \ldots + S_n + \ldots$. Introducing for convenience $x \equiv S_1 = r_1\tilde{a}(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-1}$, we find

$$\begin{aligned} \mathcal{C} &= 1 + x - (\frac{b}{d} - 1)\frac{x^2}{2} + (\frac{b}{d} - 1)(\frac{b}{d} - \frac{1}{2})\frac{x^3}{3} - (\frac{b}{d} - 1)(\frac{b}{d} - \frac{1}{2})(\frac{b}{d} - \frac{1}{3})\frac{x^4}{4} + \ldots \\ &= 1 + \frac{d}{b}(\frac{bx}{d}) + \frac{1}{2!}\frac{d}{b}(\frac{d}{b} - 1)(\frac{bx}{d})^2 + + \frac{1}{3!}\frac{d}{b}(\frac{d}{b} - 1)(\frac{d}{b} - 2)(\frac{bx}{d})^3 + \ldots \\ &= (1 + \frac{b}{d}x)^{d/b} \quad . \end{aligned} \tag{41}$$

Substituting for x yields

$$\mathcal{C} = \{1 + \frac{b}{d}[r_1\tilde{a}(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-1}]\}^{\frac{d}{b}} = [\frac{1 - \frac{b}{d}\tilde{r_1}\tilde{a} + \frac{b}{d}r_1\tilde{a}}{1 - \frac{b}{d}\tilde{r_1}\tilde{a}}]^{\frac{d}{b}} \tag{42}$$

We can write the numerator in Eq.(42) as

$$(1 - \frac{b}{d}\tilde{r_1}\tilde{a} + \frac{b}{d}r_1\tilde{a}) = [1 + \tilde{a}b(\frac{r_1 - \tilde{r_1}}{d})] = (1 + \tilde{a}L) \tag{43}$$

Where $L = b\ln(M/\mu) = b(r_1 - \tilde{r}_1)/d$. Since we are setting $c = c_2 = c_3 = \ldots = 0$ one has $(1 + \tilde{a}L)^{-1} = a/\tilde{a}$, substituting this into Eq.(42) gives

$$\mathcal{C} = [(1 - \frac{b}{d}\tilde{r_1}\tilde{a})\frac{a}{\tilde{a}}]^{\frac{-d}{b}} = [\frac{(1 - \frac{b}{d}\tilde{r_1}\tilde{a})}{\tilde{a}}a]^{\frac{-d}{b}} \tag{44}$$

Since $\tilde{a} = a(\mu) = 1/\tau$ we can rearrange Eq.(16) to obtain

$$\tilde{r_1} = \frac{d}{b}\frac{1}{\tilde{a}} - d\ln\frac{Q}{\Lambda_{\mathcal{M}}} \; , \tag{45}$$

and substituting this result into Eq.(43) we find

$$\mathcal{C} = \left(\frac{1}{b\ln(Q/\Lambda_{\mathcal{M}})}\right)^{d/b} a^{-d/b} \; . \tag{46}$$

Combining this with the anomalous dimension part $(ca)^{d/b}$ we reproduce the physical Q-dependence of $\mathcal{M}(Q)$ in Eq.(35).

5 Discussion and Conclusions

An alternative and more straightforward way of understanding the CORGI proposal is as follows. Given a dimensionless observable $\mathcal{R}(Q)$, dependent on the single dimensionful scale Q, we clearly must have, on grounds of generalized dimensional analysis [12]

$$\mathcal{R}(Q) = \Phi\left(\frac{\Lambda}{Q}\right) \; , \tag{47}$$

where Λ is a dimensionful scale, connected with the universal dimensional transmutation parameter of the theory, whose definition will depend on the way in which ultraviolet divergences are removed, $\Lambda_{\overline{MS}}$ for instance. We can try to invert Eq.(47) to obtain

$$\frac{\Lambda}{Q} = \Phi^{-1}(\mathcal{R}(Q)) \; , \tag{48}$$

where Φ^{-1} is the inverse function. This is indeed the basic motivation for Grunberg's Effective Charge approach [2, 3]. We are assuming massless quarks here. The extension if one includes masses has been discussed in [3, 13]. The structure of Φ^{-1} is [14, 15]

$$\mathcal{F}(\mathcal{R}(Q))\mathcal{G}(\mathcal{R}(Q)) = \Lambda_{\mathcal{R}}/Q \; , \tag{49}$$

where

$$\mathcal{F}(\mathcal{R}(Q)) \equiv e^{-1/b\mathcal{R}}(1+1/b\mathcal{R})^{c/b} \tag{50}$$

is a universal function of $\mathcal{R}$. $\Lambda_{\mathcal{R}}$ is connected with the universal parameter $\Lambda_{\overline{MS}}$ by the relation

$$\Lambda_{\mathcal{R}} = e^{r/b}\tilde{\Lambda}_{\overline{MS}} \;, \tag{51}$$

which follows from Eq.(11), with $r \equiv {r_1}^{\overline{MS}}(\mu = Q)$ the NLO $\overline{MS}$ coefficient. Note that r is Q-independent. The tilde over Λ reflects the convention assumed in integrating the beta-function equation to obtain Eq.(9) [6], and $\tilde{\Lambda}_{\overline{MS}} = (2c/b)^{-c/b}\Lambda_{\overline{MS}}$ in terms of the standard convention . The function $\mathcal{G}(\mathcal{R}(Q))$ has the expansion

$$\mathcal{G}(\mathcal{R}(Q)) = 1 - \frac{X_2}{b}\mathcal{R}(Q) + O(\mathcal{R}^2) + \ldots . \tag{52}$$

Here X_2 is the NNLO RS-invariant constant of integration which arises in Eqs.(13). Assembling all this we finally obtain the desired inverse relation between $\mathcal{R}$ and Λ, the universal dimensional transmutation parameter of the theory

$$Q\mathcal{F}(\mathcal{R}(Q))\mathcal{G}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \,. \tag{53}$$

Notice that all dependence on the subtraction scheme chosen resides in the single factor $e^{-r/b}$, the remainder of the expression being independent of this choice. This corresponds to the observation of Celmaster and Gonsalves [16], that Λ's with different subtraction conventions can be exactly related given a one-loop (NLO) calculation. If only a NLO calculation has been performed $\mathcal{G} = 1$ since X_2 will be unknown, so that the best one can do in reconstructing $\Lambda_{\overline{MS}}$ is

$$Q\mathcal{F}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \,. \tag{54}$$

This is precisely the result obtained on inverting the NLO CORGI result $\mathcal{R} = a_0$ given by Eq.(21).

The essential point is that the dimensional transmutation scale Λ is the fundamental object. In contrast the convention-dependent dimensionful scales μ and M are ultimately irrelevant quantities which cancel out of physical predictions if one takes care to resum *all* of the ultraviolet logarithms that build the physical Q-dependence in association with Λ. Our purpose

has been to indicate that the unphysical μ and M dependence of conventional fixed-order perturbation theory reflects its failure to resum all of these RG-predictable terms. We have analyzed how Eq.(54) is built by explicitly resumming the convention-dependent logarithms together with the ultraviolet logarithms. Having done this, however, one can simply use Eq.(53) to test perturbative QCD. Given at least a NLO calculation for an observable $\mathcal{R}(Q)$ one simply substitutes the data values into Eq.(53), where $\mathcal{G}(\mathcal{R}(\mathcal{Q}))$ can include NNLO and higher corrections if known, and obtains $\Lambda_{\overline{MS}}$. To the extent that remaining higher-order perturbative and possible power corrections are small, one should find consistent values of $\Lambda_{\overline{MS}}$ for different observables. There is no need to mention μ or M in this analysis, let alone to vary them over an *ad hoc* range of values. For the moment problem the result coresponding to Eq.(53) is

$$Q\overline{\mathcal{F}}\left(\frac{\mathcal{M}}{A}\right)\overline{\mathcal{G}}\left(\frac{\mathcal{M}}{A}\right)e^{-\hat{r}/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}}\ , \tag{55}$$

where $\hat{r}{\equiv}\hat{r}_1^{\overline{MS}}(\mu = Q)$ is defined in Eq.(26). The modified functions $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are most easily obtained by noting that $\hat{\mathcal{R}}$ in Eq.(24) is directly related to $\mathcal{M}/A$ and also satisfies Eq.(53). One finds

$$\begin{aligned}\overline{\mathcal{F}}(x) &= \exp[bc(1-x^{-b/d}](1+bc(x^{-b/d}-1))^{c/b}\\ \overline{\mathcal{G}}(x) &= \left(1-\frac{X_2}{d}\frac{x^{b/d}}{c(1-x^{b/d})}+\ldots\right)\ .\end{aligned} \tag{56}$$

Where X_2 is the NNLO FRS-invariant which arises in Eqs.(18). The scheme-independent parameter A reflects a physical property of the operator $\mathcal{O}_n$ in Eq.(2). A_n and $\Lambda_{\overline{MS}}$ should be fitted simultaneously to the data for $\mathcal{M}_n(Q)$ using Eq.(55).

We hope to report direct fits of data to $\Lambda_{\overline{MS}}$ as outlined above, for both e^+e^- jet observables [17] and structure functions and their moments [18], in future publications.

Aknowledgements

A.M. acknowledges the financial support of the Iranian government and also thanks S.J. Burby for useful discussions.

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